

Last time classified disc. subgps of $(\mathbb{R}^2, +)$

Can have

L

- $L = \{0\}$

- $L = \mathbb{Z}a, \quad a \neq 0 \quad a \in \mathbb{R}^2$

- $L = \mathbb{Z}a + \mathbb{Z}b$ with a, b a basis of $\mathbb{R}^2$

Trying to understand $G \leq M_2$ that are discrete

$$G \rightsquigarrow L = \{v \in \mathbb{R}^2 : t_v \in G\} \quad \text{a disc subgp of } (\mathbb{R}^2, +)$$

$$\begin{array}{ccc} G & \rightsquigarrow & \bar{G} : \varphi : M_2 \rightarrow O_2 \\ & & \parallel \\ & & t_a p \mapsto p, \quad \begin{array}{l} a \in \mathbb{R}^2 \\ p \in O_2 \end{array} \\ & & \vdots \\ & & \varphi(G) \end{array}$$

$\bar{G}$ is a discrete subgp of O_2

Propn A disc. subgp $\bar{G}$ of O_2 is nec. finite, and thus either cyclic or dihedral.

Pf: First, set $H = \bar{G} \cap SO_2$.

- Define $\Gamma = \{t \in \mathbb{R} : p_t \in H\}$.

- $\Gamma \leq (\mathbb{R}, +)$, and is discrete

$$\Rightarrow \Gamma = \mathbb{Z}a \text{ for some } a \in \mathbb{R}$$

- $2\pi \in \Gamma \Rightarrow 2\pi = n \cdot a \Rightarrow a = \frac{2\pi}{n}$

$\Rightarrow \Gamma = \mathbb{Z}(\frac{2\pi}{n}) \Rightarrow H \cong C_n$ so is finite.

Either $\bar{G} = \bar{G} \cap SO_2$ or $\exists r' \in \bar{G} \setminus H$.

Then, every elt of $\bar{G}$ is either in H or in Hr' .

(Pf: if $g \in \bar{G} \setminus H$, then $g(r')^{-1}$ has det 1
 $\Rightarrow g(r')^{-1} \in H \Rightarrow g \in Hr'$)

Propⁿ $G \subseteq M$ is a subg, $a \in L$ so that $t_a \in G$, and

$\bar{g} \in \bar{G} \subseteq O_2$. Then $\bar{g}(a) \in L$.

Pf: $\bar{g} \in \bar{G} \Rightarrow \exists b \in \mathbb{R}^2$ so that $g = t_b \bar{g} \in G$.

$$g t_a g^{-1} = t_b \bar{g} t_a (\bar{g})^{-1} t_{-b}$$

$$= t_b t_{\bar{g}(a)} t_{-b}$$

$$= t_{\bar{g}(a)} \in G$$

$$\Rightarrow \bar{g}(a) \in L$$

Thm (Crystallographic Restriction) Suppose $L \subseteq (\mathbb{R}^2, +)$ is discrete and $\neq \{0\}$. Suppose $H \subseteq O_2$ satisfies

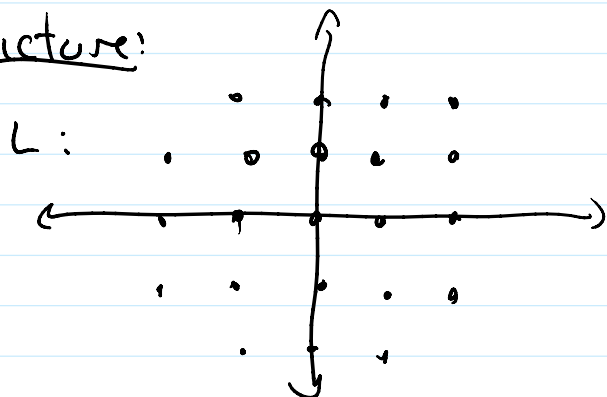
$h(L) \subseteq L \quad \forall h \in H$ and $g \in L$. Then

$H \cong C_n$ or D_n with $n=1, 2, 3, 4, 6$.

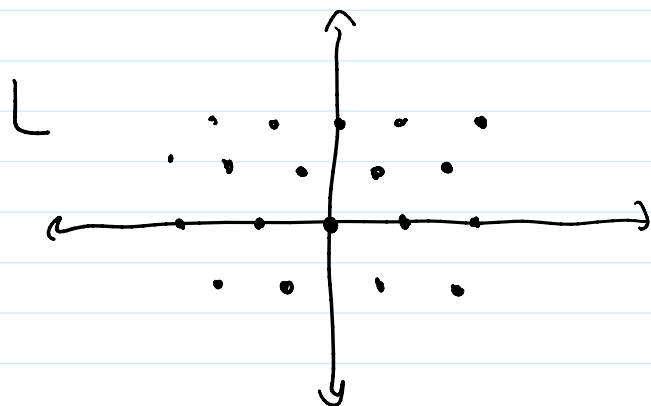
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Consequently, if $G \subseteq M_2$ is discrete with $L \neq \{0\}$, $\bar{G}$ is isom to C_n or D_n with $n=1, 2, 3, 4$ or 6 .

Picture:



$$\bar{G} = C_4 \text{ or } D_4$$

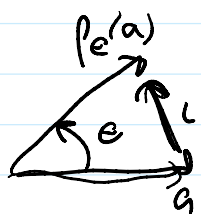


$$\bar{G} = C_6 \text{ or } D_6 \\ \text{or } C_3 \text{ or } D_3$$

Pf: Suffices to show that if $p_e \in H$ then p_e has order 1, 2, 3, 4 or 6.

Take $a \in L$ to be $\neq 0$ and of min length.

$$a \in L \Rightarrow p_e(a) \in L \text{ if } p_e \in \bar{G}.$$



$$p_e(a) - a \in L.$$

$$\text{If } \theta < \frac{2\pi}{6} \text{ then } \|p_e(a) - a\| < \|a\|$$

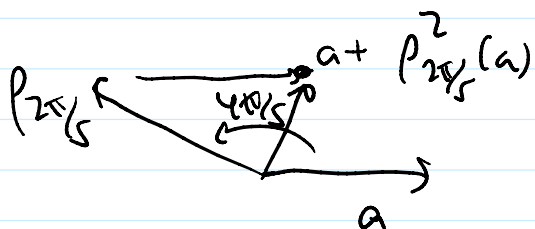
$$\Rightarrow 12 > 2\pi$$

$$e^{i\theta} = \frac{a}{b} \quad \text{then} \quad \|e^{i\theta} - 1\| = \frac{2}{b} \sin \frac{\theta}{2}$$

$$\Rightarrow \theta \geq \frac{2\pi}{b}$$

$\Rightarrow \rho_e$ has order 1, 2, 3, 4, ~~5~~ or 6

must rule this out



$$\|a + p_{4\pi/5}(a)\| < \|a\|.$$

QED

Group Actions (Group operations)

G a gp. S a set.

A gp action is a map $G \times S \rightarrow S$, denoted $(g, s) \mapsto g \cdot s$ satisfying

$$\bullet 1 \cdot s = s$$

$$\bullet g_1 \cdot (g_2 \cdot s) = (g_1 g_2) \cdot s$$

$$\forall s \in S \text{ and } \forall g_1, g_2 \in G.$$

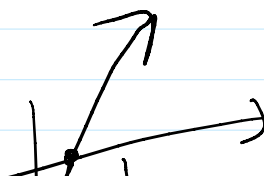
Group actions abound We will $G \ni S$.

$$\bullet S_n \ni \{1, 2, \dots, n\}$$

$$p \cdot s = p(s)$$

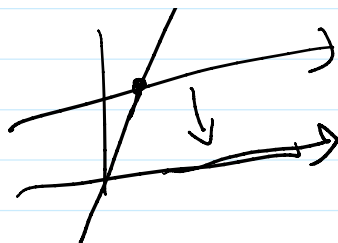
$$\bullet O_n \ni \mathbb{R}^n, \quad GL_n(\mathbb{R}) \ni \mathbb{R}^n$$

$$\bullet M_n \ni \mathbb{R}^n$$



- $M_n \subset \mathbb{R}^n$

- $M_2 \subset \{\text{line in plane}\}$



- $G \subset G/H$
set of cosets $g \cdot (g'H) = gg'H.$

Suppose $g \in G$. Let $m_g: S \rightarrow S$ that is
 $m_g(s) = g * s.$

Then m_g is a permutation of S , i.e. a bijection

PF: Consider $m_{g^{-1}}$. Then $m_{g^{-1}} \circ m_g = m_{g^{-1}g} = m_{id} = Id_S$

similarly, $m_g \circ m_{g^{-1}} = Id_S$

$\Rightarrow m_{g^{-1}}$ is the inverse fun to $m_g \Rightarrow m_g$ is a bijection.
